Toward a Consistent Beam Theory

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It is well known that the Euler-Bernoulli theory of the bending of beams makes use of a contradicting assumption of zero shear strains and nonzero shear stresses. Sometimes, this type of assumption is also carried over to more refined shear deformation theories. This paper outlines a theory that avoids this assumption. With the aid of the specific example of a tip-loaded cantilever beam, it is shown that the present theory gives Euler-Bernoulli solutions in that part of the beam where shear deformation is unimportant and a shear deformation type of solution in the part of the beam where shear deformation is important, with transition stress patterns between the two. Numerical studies, with a shear modulus representative of sandwich beams, bring out the usefulness of the present theory for the analysis of such soft-cored beams.

Introduction

THE Euler-Bernoulli theory of bending is widely used for the analysis of beams. In this theory, the transverse shear strain is zero and the transverse shear stress is nonzero. It is difficult to justify this assumption. This situation is accepted by assuming that the beam behaves as if its shear modulus is infinity. This theory quite satisfactorily predicts the behavior of slender beams throughout the domain of the beam, except at supports. When the beam is short and/or of shear flexible construction, the theory needs some modification to include the effect of transverse shear; such a modification is called a shear deformation theory. The shear deformation effect can be included by considering the transverse displacement as a sum of two partial displacements, one due to bending and the other to shear. It is assumed that the process of shearing is such that the shear strain, and thus the shear stress across the cross section, is constant. Recognizing the need for more refined theoretical models, a higher order theory for short beams has been introduced in Refs. 1 and 2. This theory is in the form of a hierarchy of sets of governing equations with each set describing the beam behavior to a certain degree of approximation. The zeroth-order approximation of this formulation corresponds to the Euler-Bernoulli theory (EBT) the elementary theory of bending. The first-order approximation corresponds to the Timoshenko's shear deformation theory (SDT). In Refs. 1 and 2, it has been shown that an attempt to improve the performance of the SDT by using refined values of the shear coefficient leads to greater discrepancies between the predictions of the SDT and the values estimated by the higher-order theory. Thus, there is a need to consider higher-order models.

In Ref. 3 an attempt has been made to formulate a beam theory that includes all of the three important secondary effects—the transverse shear, the nonclassical axial stress, and the normal strain effects. In the development of this theory, guidance has been drawn from the elasticity solution of a uniformly loaded, simply supported beam. This theory has given excellent results for the cases considered. As the choice of the normal stress distribution is necessarily influenced by the normal stress/strain condition at the surface of the beam, any single choice of the normal strain distribution across the depth of the beam that includes the normal strain effect is unlikely to be satisfactory for all types of loading conditions. The presence of discontinuous loading poses additional problems. In particular, when the beam is partially loaded and/or has concentrated loads, the choice of normal strain distribution is very complicated. This complicating effect is not considered in this paper. A theory that includes the transverse shear strain and nonclassical axial stress and that can consider any arbitrary transverse loading condition is given in this paper. This theory corresponds to the second-order approximation of Ref. 2. However, the present formulation is an improvement over the formulation given in Ref. 2, in the sense the present formulation includes the coupling effect between the nonclassical axial stress and the shear, not considered in Ref. 2.

In Ref. 4, a procedure has been outlined for obtaining improved shear stress distribution across the depth of the beam by integrating equations of equilibrium, as is usually done in the elementary theory of bending. This procedure involves violation of the stress-strain relationship. The present formulation is an attempt to obtain the standard parabolic variation of the transverse shear stress across the depth of the beam without violating the stress-strain relationship.

A cantilevered short beam with length-to-depth ratio of 2.5 is considered for detailed analysis. Displacement distributions along the span and stress distributions across the depth have been determined using the SDT and the present theory. The results indicate that the SDT is inadequate for predicting the shear stress distributions as it gives a constant shear stress across the depth, contrary to the general pattern of a combination of constant and parabolic distributions. The present theory gives stress distributions similar to the SDT near the fixed end and similar to EBT away from the fixed end, with transition patterns between the two. Thus, the present theory gives appropriate stress distributions throughout the beam without violating the stress-strain relationship.

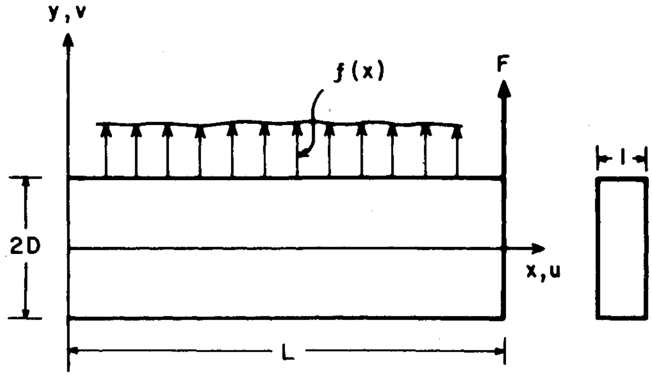

Fig. 1 A typical beam.

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Formulation

Figure 1 shows a typical beam, along with the coordinate system employed. The beam has uniform rectangular cross section with depth $2D$ and width unity. It is made of an isotropic material with Young's modulus E and shear modulus G. The span of the beam is L.

Following Refs. 1 and 2, the transverse deflection V is considered as a sum of two partial deflections, the deflection due to bending V_B and the deflection due to transverse shearing V_S. Thus

$$V(x) = V_B(x) + V_S(x) \tag{1}$$

The in-plane displacement u at any point in the beam may be written as[1,2]

$$u = -yV_{B,x} - p(y)\phi(x) \tag{2}$$

where $,x$ indicates the derivative with respect to x. The first term is due to rotation of the cross section. $p(y)$ describes the variation of the second component of u in the cross section and is chosen such that the shear strain due to this term is zero at the free edges ($y=\pm D$) and is maximum at $y=0$. More discussion on the choice of this function is given in Refs. 1 and 2. $p(y)$ is taken here as

$$p = y[1 - \tfrac{1}{3}(y^2/D^2)] \tag{3}$$

The state of strain in the beam is described in terms of two strains ϵ_x and ϵ_{xy} given by

$$\epsilon_x = \frac{\partial u}{\partial x} = -yV_{B,xx} - p\phi_{,x} \tag{4a}$$

$$\epsilon_{xy} = \frac{\partial u}{\partial y} + \frac{\mathrm{d}v}{\mathrm{d}x} = V_{S,x} - p_{,y}\phi \tag{4b}$$

where $,y$ indicates the first derivative with respect to y. The state of stress is described by σ_x and σ_{xy}, which are related to the strains as

$$\sigma_x = E\epsilon_x \tag{5a}$$

$$\sigma_{xy} = G\epsilon_{xy} \tag{5b}$$

Using Eqs. (1-5), expressions for the strain energy S and the work of external loads W may be written as

$$S = \frac{1}{2}\int(\sigma_x\epsilon_x + \sigma_{xy}\epsilon_{xy})\,\mathrm{d}A\,\mathrm{d}x$$

and

$$W = \int f(x)\,v(x)\,\mathrm{d}x$$

where $f(x)$ is the applied transverse loading on the beam. According to the principle of virtual displacements, when the beam is in a state of equilibrium, the first variation of the total potential energy π will be zero, or

$$\delta\pi = \delta(S - W) = 0$$

Utilizing this principle, the governing equations of equilibrium and the boundary conditions can be established as follows:

Governing equations

$$B_0 V_B'''' + b\phi''' = f \tag{6a}$$

$$S_0 V_S'' - s\phi' = -f \tag{6b}$$

$$B\phi'' - S\phi + bV_B''' + sV_S' = 0 \tag{6c}$$

Boundary conditions

Fixed end:

$$V_B = V_S = V_B' = \phi = 0 \tag{7}$$

Free end:

$$\phi' = V_B'' = 0 \tag{8a}$$

$$S_0 V_S' - s\phi = F \tag{8b}$$

$$B_0 V_B''' + b\phi'' = -F \tag{8c}$$

where F is the concentrated load at the (boundary) end of the beam. It is assumed that the beam is not subjected to any concentrated bending moments. Primes indicate the order of the derivative.

The cross-sectional constants in Eqs. (6-8) are given by

$$B_0 = \int_A Ey^2\,\mathrm{d}A = \frac{2}{3}ED^3 \tag{9a}$$

$$B = \int_A Ep^2\,\mathrm{d}A = \frac{136}{315}ED^3 \tag{9b}$$

$$b = \int_A Eyp\,\mathrm{d}A = \frac{8}{15}ED^3 \tag{9c}$$

$$S_0 = \int_A G\,\mathrm{d}A = 2GD \tag{9d}$$

$$S = \int_A Gp_{,y}^2\,\mathrm{d}A = \frac{16}{15}GD \tag{9e}$$

$$s = \int_A Gp_{,y}\,\mathrm{d}A = \frac{4}{3}GD \tag{9f}$$

Table 1 Partial deflections ($E=6.9\times10^{10}$, $G=2.6\times10^{10}$, $L=0.5$, $L/D=5.0$)

x/L	V_B (SDT) $\times10^9$	V_B (PT) $\times10^9$	V_S (SDT) $\times10^{10}$	V_S (PT) $\times10^{10}$	V_T (SDT) $\times10^9$	V_T (PT) $\times10^9$
0	0	0	0	0	0	0
0.1	0.105	0.122	0.192	0.0516	0.124	0.127
0.2	0.406	0.446	0.385	0.0529	0.444	0.451
0.3	0.880	0.943	0.577	0.053	0.938	0.949
0.4	1.51	1.59	0.769	0.053	1.58	1.60
0.5	2.26	2.37	0.962	0.053	2.36	2.38
0.6	3.13	3.26	1.15	0.053	3.25	3.27
0.7	4.08	4.24	1.35	0.053	4.22	4.24
0.8	5.10	5.28	1.54	0.053	5.26	5.29
0.9	6.16	6.36	1.73	0.053	6.34	6.37
1.0	7.25	7.47	1.92	0.053	7.44	7.48

Cantilever Beam under Tip Load

The solution for the deflection of a cantilevered beam under a unit tip load can be easily worked out as

$$V_B = d_0 + d_1 x + d_2 x^2 + d_3 x^3 - \frac{b}{\lambda B_0}(\bar{A}\cosh\lambda x + \bar{B}\sinh\lambda x) \quad (10)$$

$$V_S = a_0 + a_1 x + \frac{s}{S_0}(\bar{A}\cosh\lambda x + \bar{B}\sinh\lambda x) \quad (11)$$

$$\phi = \bar{A}\sinh\lambda x + \bar{B}\cosh\lambda x + \bar{C} \quad (12)$$

where

$$\lambda = \sqrt{(S - s^2/S_0)/(B - b^2/B_0)} \quad (13)$$

$$\bar{A} = -\bar{B}\tanh\lambda L \quad (14a)$$

$$\bar{B} = F\left(\frac{b/B_0 - s/S_0}{S - s^2/S_0}\right) \quad (14b)$$

$$\bar{C} = -\bar{B} \quad (14c)$$

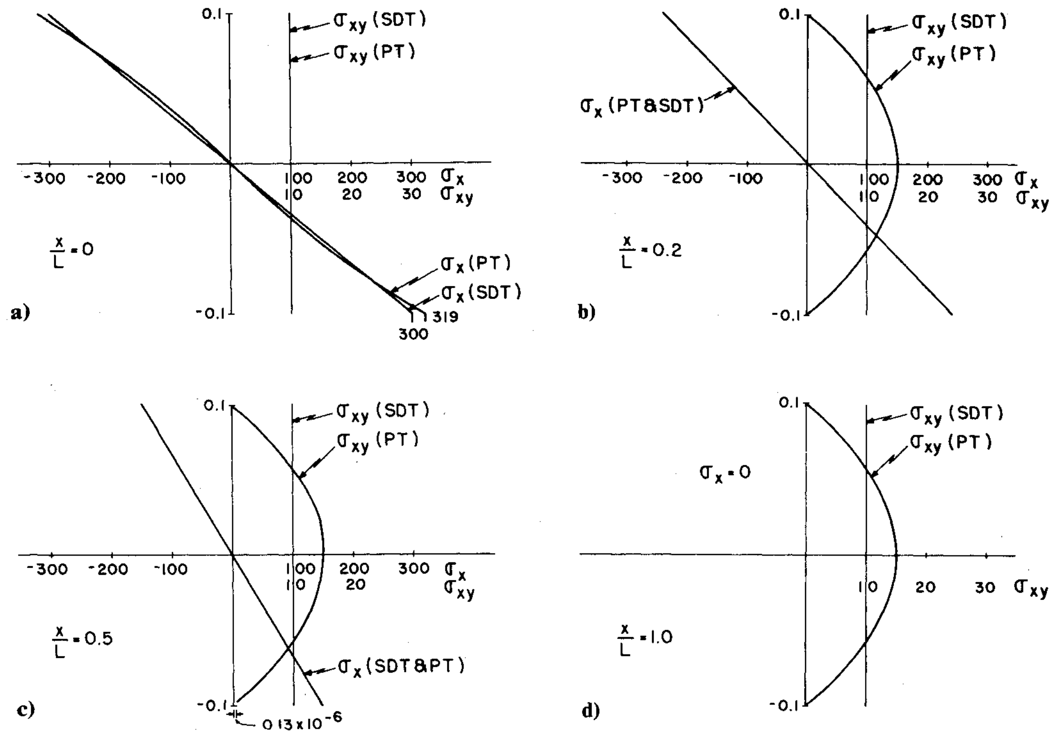

Fig. 2 Stress distribution.

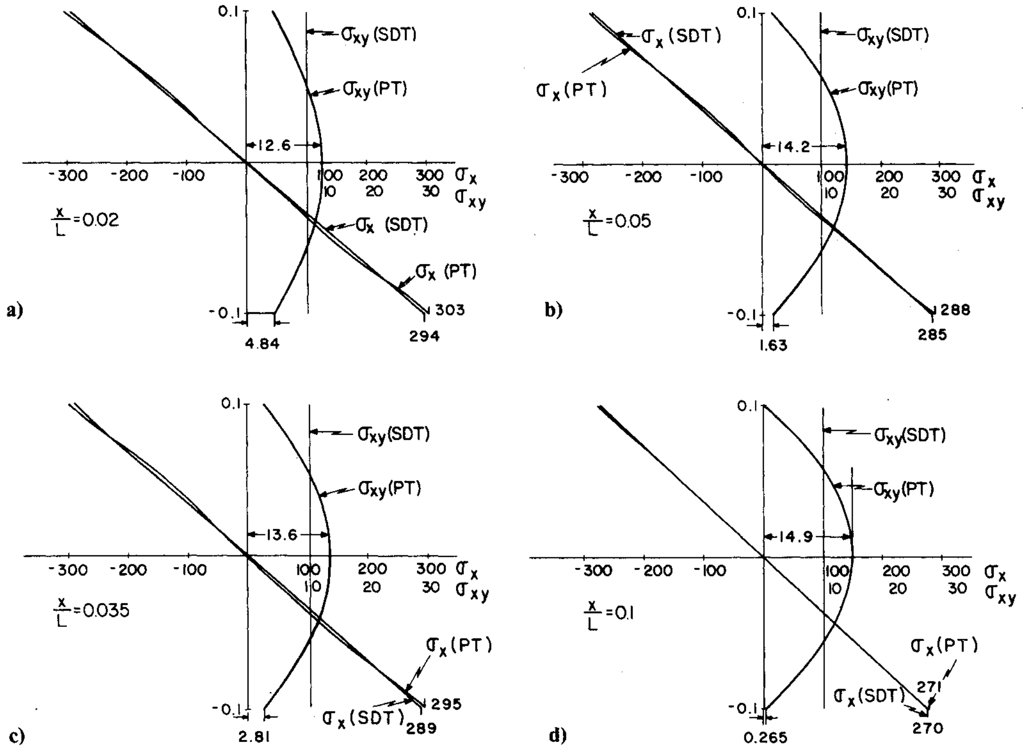

Fig. 3 Transition stress pattern.

$$a_0 = \bar{B}\frac{s}{\lambda S_0}\tanh\lambda L \tag{15a}$$

$$a_1 = \frac{F}{S_0} - \frac{s}{S_0}\bar{B} \tag{15b}$$

$$d_0 = -\bar{B}\frac{b}{\lambda B_0}\tanh\lambda L \tag{16a}$$

$$d_1 = \bar{B}\frac{b}{B_0} \tag{16b}$$

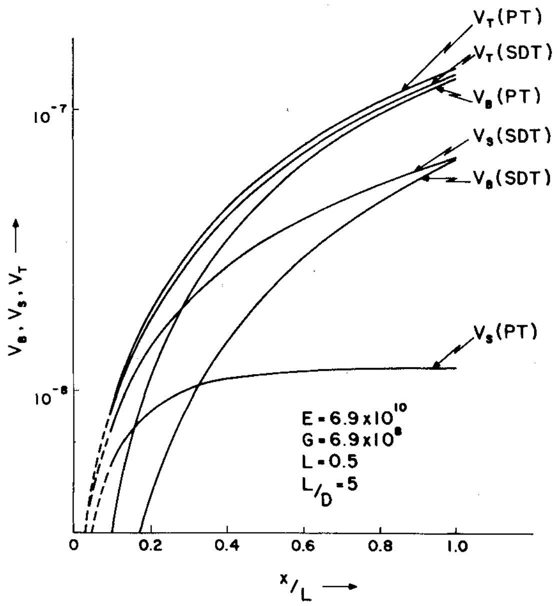

Fig. 4 Comparison of partial deflections for a short beam.

$$d_2 = F\frac{L}{2B_0} \tag{16c}$$

$$d_3 = -F\frac{l}{6B_0} \tag{16d}$$

The solution of the SDT is a special case of this solution and is obtained by setting $\bar{B}=0$. The EBT solution is obtained by setting $\bar{B}$ and a_1 equal to zero.

Discussion of the Results

Numerical results for the deflections and the stresses have been obtained by taking the data corresponding to an aluminum alloy beam: $L=0.5$ m, $L/D=5$, $E=6.9\times10^{10}$ N/m^2 and $G=2.6\times10^{10}$N/m^2.

Table 1 provides a comparison of partial deflections V_B and V_S and the total deflection V_T as estimated by the SDT and the present theory (PT). The difference in the total deflection between SDT and PT is small. But there are significant differences in the partial deflections. V_S is considerably overestimated, whereas V_B is underestimated by SDT.

The stress distributions across the depth of the beam are shown in Fig. 2. Figure 2a gives the stress distribution at the fixed end of the beam. The σ_x distribution is different from the classical linear distribution. The SDT predicts a maximum axial stress of 300 N/m^2, whereas the PT gives 319 N/m^2, i.e., about 6.7% higher value. But the shear stress distribution by the PT is the same as the SDT. A perusal of stress patterns at $x/L=0.2$ (Fig. 2b), $x/L=0.5$ (Fig. 2c), and $x/L=1$ (Fig. 2d) shows that the shear stress distribution is of parabolic type. Beyond $x/L=0.5$, the constant part of the shear stress is less than 0.13×10^{-6} N/m^2. The σ_x distribution is almost the same as the SDT.

Figure 3 shows the transition pattern of stress distribution at cross sections near the fixed end. The σ_x distribution changes from a cubic type at the fixed end to a linear type, whereas the shear stress distribution changes from a constant

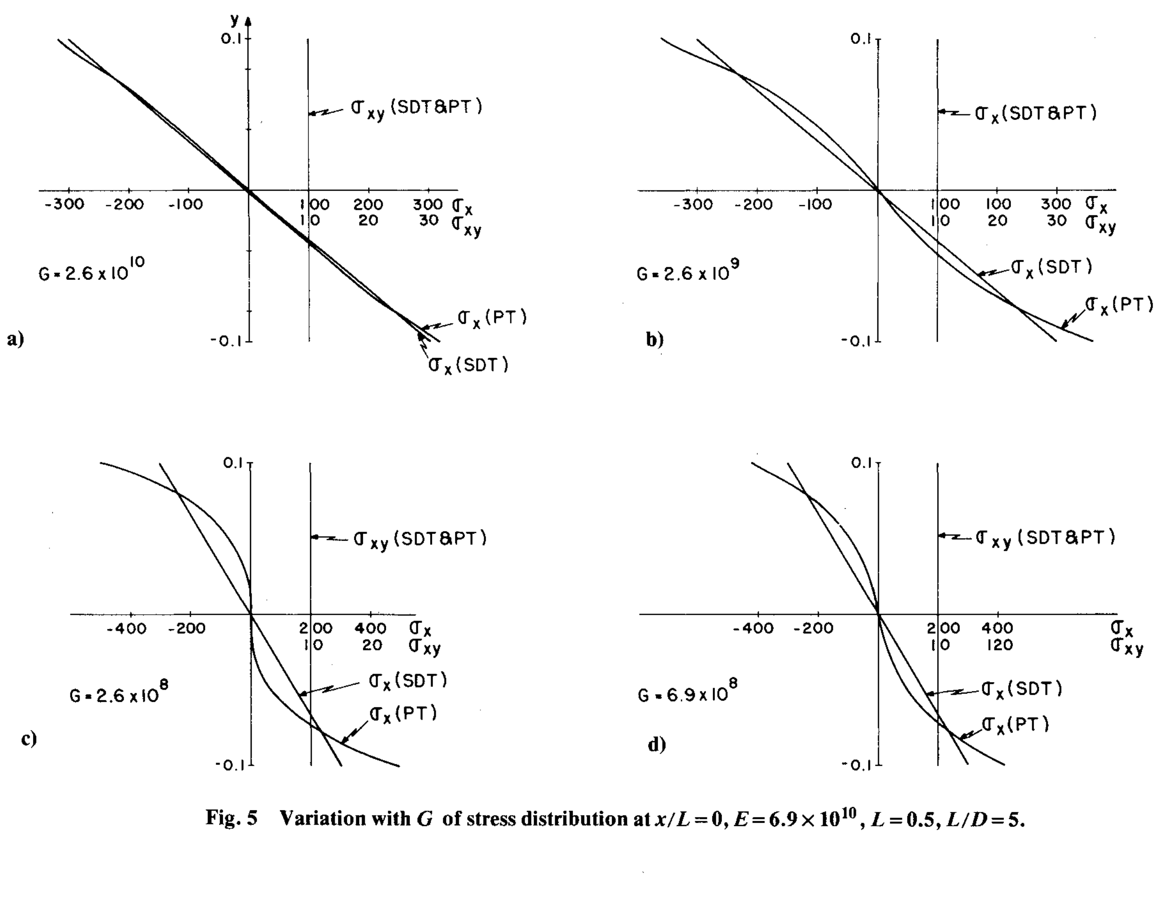

Fig. 5 Variation with G of stress distribution at $x/L=0$, $E=6.9\times10^{10}$, $L=0.5$, $L/D=5$.

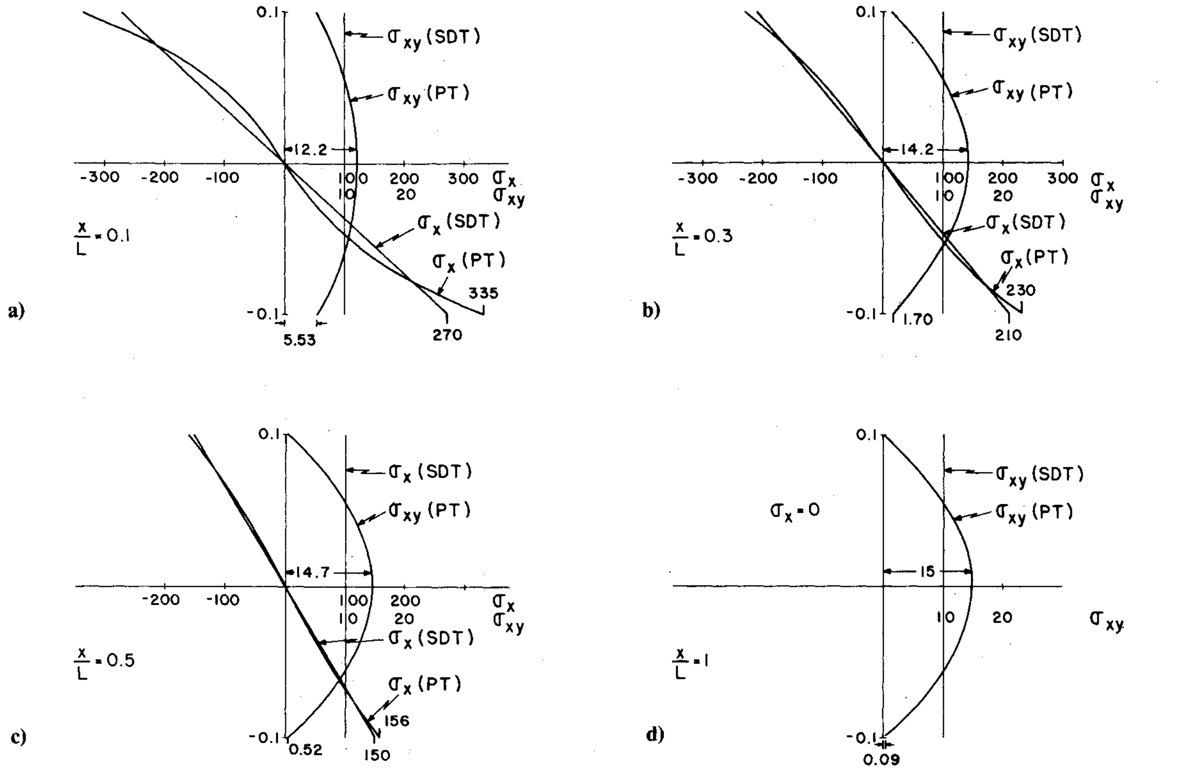

Fig. 6 Stress distribution in the soft beam $G = 6.9 \times 10^8$.

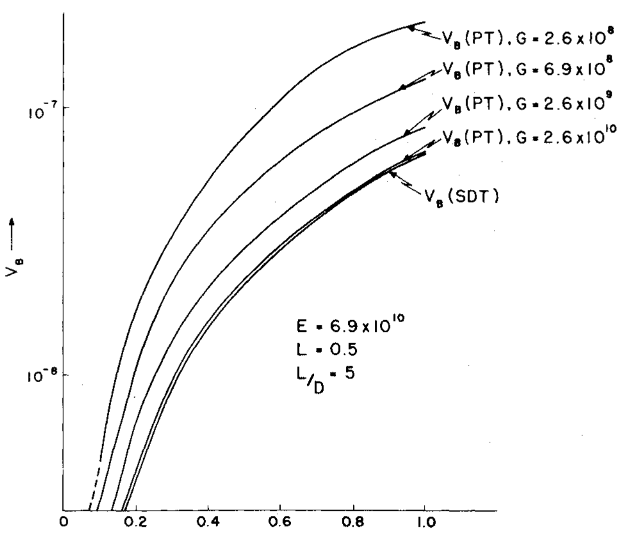

Fig. 7 Spanwise variation of V_B.

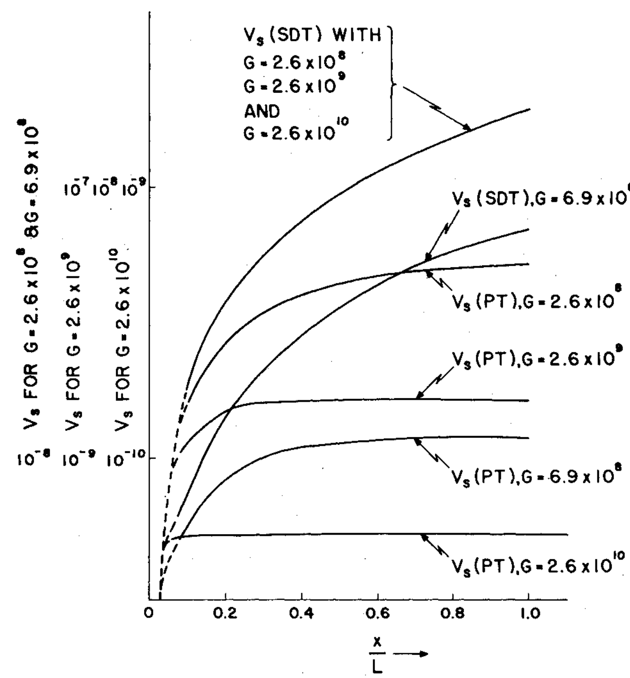

Fig. 8 Spanwise variation of V_S.

type at the fixed end to a parabolic type. The transition is almost complete at $x/L = 0.1$.

Numerical results have been obtained by considering a lower value of the shear modulus to examine the performance of the PT in the analysis of soft beams such as sandwich beams. A value of $G = 6.9 \times 10^8$ N/m^2, which corresponds to an aluminum honeycomb, is chosen for the numerical work. Figure 4 shows a comparison of the partial deflections. The shear deflection V_S is considerably overestimated and the bending deflection V_B is underestimated by the SDT. Figures 5d and 6 show the stress distributions at various cross sections of the soft beam. At the fixed end, the SDT gives a value of 300 N/m^2 for the maximum axial stress, whereas the PT estimate is 418 Pa. The deviation of the axial stress from the EBT or SDT, which is usually called nonclassical axial stress, can be as large as 40% in soft beams. The region in which this nonclassical axial stress is significant is much larger in soft beams than a similar region in corresponding isotropic solid beams. In the present case, the nonclassical axial stress is significant even at $x/L = 0.3$ (Fig. 6b). The shear stress distribution is a combination of the constant and parabolic distributions and this pattern continues throughout the beam.

Table 2 gives a comparison of total deflection for three values of the shear modulus. Figure 5 shows the change in the

Table 2 Total deflections

x/L	$G=2.9\times10^8$ V_T (SDT) $\times10^8$	$G=2.9\times10^8$ V_T (PT) $\times10^8$	$G=2.9\times10^9$ V_T (SDT) $\times10^8$	$G=2.9\times10^9$ V_T (PT) $\times10^8$	$G=2.9\times10^{10}$ V_T (SDT) $\times10^8$	$G=2.9\times10^{10}$ V_T (PT) $\times10^8$
0	0	0	0	0	0	0
0.1	0.203	0.209	0.030	0.031	0.012	0.013
0.2	0.425	0.447	0.079	0.084	0.044	0.045
0.3	0.665	0.710	0.146	0.154	0.094	0.095
0.4	0.920	0.992	0.228	0.240	0.158	0.160
0.5	1.19	1.29	0.323	0.338	0.236	0.238
0.6	1.47	1.60	0.428	0.448	0.325	0.327
0.7	1.75	1.93	0.543	0.567	0.422	0.424
0.8	2.05	2.25	0.664	0.691	0.526	0.529
0.9	2.35	2.59	0.789	0.821	0.634	0.637
1.0	2.65	2.93	0.917	0.952	0.744	0.748

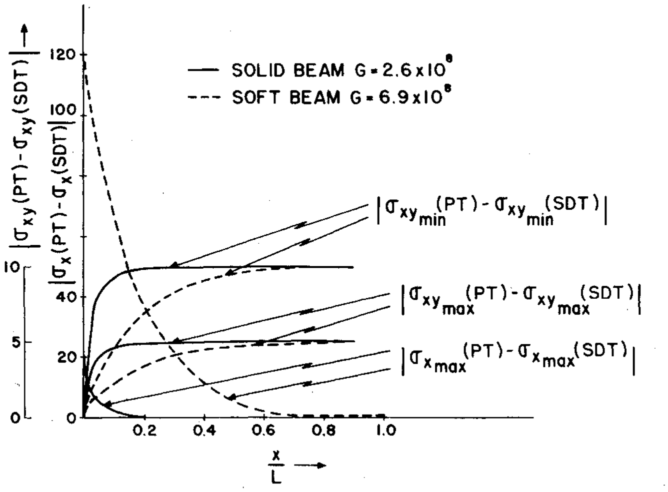

Fig. 9 Differences in stresses by SDT and PT.

stress distribution pattern at the fixed end of the beam. Figures 7 and 8 show the spanwise distribution of the partial deflections. These results show that the deviation between the STD and the PT increases as the shear modulus becomes smaller. Figure 9 shows the variation in the moduli of the differences in the maximum axial stress, maximum shear stress, and minimum shear stress by PT and SDT. For solid beams the deviations are confined to small region near the fixed end $0\leq x\leq 0.2L$, whereas for soft beams these deviations are significant throughout the beam. This highlights the need for this theory in the analysis of sandwich beams.

Conclusions

This paper gives an improved theory for the bending analysis of beams. This formulation is in the form of three simultaneous ordinary differential equations as compared to a single fourth-order equation in the Euler-Bernoulli theory and two ordinary differential equations in the shear deformation theory, both of which are special cases of the present formulation.

The stress and displacement distributions in a typical beam are obtained using the present theory and the shear deformation theory. The present theory has given appropriate stress and displacement distributions at all sections in the span of the beam. Study of the results for beams with lower shear rigidity indicated the usefulness of this theory for sandwich beams. As the elementary theory of bending and the shear deformation theory are special cases of the present theory and as this formulation gives appropriate stress distribution throughout the domain of the beam, the present theory can be adopted as a basis for the formulation of consistent plate and shell theories.

It may be noted here that the provision for the constant shear stress across the depth of the beam, as outlined in the present paper, is not mandatory. A theory without this type of constant shear stress can easily be generated as a special case of the formulation given in this paper. Such a theory will give parabolic shear stress variation across the depth of the beam at all sections along the span of the beam, including the fixed end. In the past, Timoshenko, Mindlin, and several others have considered a constant variation of transverse shear. The present study also indicated that in a region near the fixed end the transverse shear stress variation is almost constant. Further investigation is necessary to confirm the need to include or to discard the term corresponding to the constant variation of shear stress in the formulation.

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